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A Mixed Problem for Navier-Stokes System

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Abstract. Mixed problems are changed to boundary value problems by making used of Laplace transform. In classical boundary value problems, boundary conditions are local, but boundary conditions could be global ([2, 3, 11]). Every boundary value problem depends on a potential method in mathematical-physics theory. Of course, one couldn't solve some many problems by potential methods. We shall give a method in which one could reduce every boundary value problem to the second kind Fredholm integral equations and then solve it. In fact, we could obtain solution of every local, non-local or global boundary value problem by this method. Finally, we shall give some sufficient conditions for existence of solutions of the Fefferman's problem A ([4]).

1. Introduction

There are usually mathematical models based on differential equations, integral equations and integro-differential equations for physical and natural events. These models are frequently based on Cauchy problem, boundary value problem or mixed problem ([5-9, 18-20]). If there is the time variable in these equations, for verification of their solutions we obtain a boundary value problem which is depend on complex parameter by using the method in [15] or Laplace transform (see [7, 18]).

The boundary value problem may be in a bounded or unbounded region. We must provide boundary conditions in bounded regions, but solutions of the problems and their derivatives are periodic or tend to zero at infinity in unbounded regions.

The potential theory is useful in many boundary value problems, for example Dirichlet and Neumann problems. But, potential theory is not efficient in solving of many another problems (see [1-3, 10, 12-14, 17]). We shall give a method that it will be efficient than the potential theory.

The Euler and Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ (n=2 or n=3). These equations are to be solved for an unknown velocity

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vector $u(x,t) = (u_i(x,t))_{1\leq i\leq n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \geq 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are then given by

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), \qquad (x \in \mathbb{R}^n, t \geq 0) \tag{1}$$

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0, \qquad (x \in \mathbb{R}^n, t \geq 0) \tag{2}$$

with initial conditions

$$u(x,0) = u^0(x), \qquad (x \in \mathbb{R}^n). \tag{3}$$

Here, $u^0(x)$ is a given C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given externally applied force (e.g. gravity), ν is a positive coefficient (the viscosity) and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables. The Euler equations are equations (1), (2), (3) with ν is equal to zero.

Equation (1) is just Newton's low $f = ma$ for a fluid element subject to the external force $f = (f_i(x,t))_{1\leq i\leq n}$ and to the forces arising from pressure and friction. Equation (2) just says that fluid is incompressible. For physically reasonable solutions, we want to make sure $u(x,t)$ does now grow large as $|x| \longrightarrow \infty$. Hence, we will restrict to forces f and initial conditions u^0 that satisfy

$$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K} \tag{4}$$

on $\mathbb{R}^n$, for any α and K, and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \tag{5}$$

on $\mathbb{R}^n \times [0,\infty)$, for any α, m, K.

We accept a solution of (1), (2) and (3) as physically reasonable only if it satisfies

$$p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{6}$$

and

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \,\mathrm{d}\, x < C, \quad \text{for all} \quad t \geq 0 \quad \text{(bounded energy)}. \tag{7}$$

A fundamental problem in analysis is to decide whether such smooth, physically reasonable solutions exist for the Navier-Stokes equations. To give reasonable leeway to solvers while retaining the heart of the problem, Fefferman has provided four problems A, B, C and D. The authors have verified the problems C and D in [4]. Here, we restate the problem A.

(A) Existence and Smoothness of Navier-Stokes Solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^0(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6) and (7).

Definition 1.1. A bounded function $f : (a,b) \to \mathbb{R}$ is called regular if $f(x) = \frac{f(x^+)+f(x^-)}{2}$ for all $x \in (a,b)$. If I is a closed or half-open interval, then a bounded function $f : I \to \mathbb{R}$ is called regular whenever $h : \mathbb{R} \longrightarrow \mathbb{R}$ so is, where

$$h(x) = \begin{cases} f(x), & x \in I \\ 0, & \text{elsewhere.} \end{cases}$$

Let $f : (a,b) \longrightarrow \mathbb{R}$ be a bounded function and let the set $E = \{x \in (a,b) : f \text{ is not defined at } x \text{ or } f \text{ is not continuous at } x\}$ be countable. Then, there is an unique bounded regular function $g : (a,b) \longrightarrow \mathbb{R}$ such that $g|_{(a,b)\setminus E} = f$.

We call $x = a$, $\frac{m}{n}$-th repeated root of $\sqrt[n]{(x-a)^m} = 0$ as we called $x = a$, n-th repeated root of $(x-a)^n = 0$. In general, we call $x = a$, α-th repeated root of $(x-a)^\alpha = 0$ for all real $\alpha > 0$.

It is well known in references that, α is singularity of order $-\alpha$ for the function $f(x) = x^\alpha$, when $\alpha < 0$ and

$$\int_{-\infty}^{x} \delta(t)\, \mathrm{d}\, t = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0, \end{cases}$$

where δ is the Dirac's function.

It be differentiated of the equation $x\delta(x) = 0$ in some mathematical works (for example, [7] and [19]), but we do not accept it in this paper and we agree it while there does not exist differentiation action. We have a similar sight on the equation $|x|^2 = x^2$ (x is real).

As we know in potential theory,

$$U(x-\xi, t-\tau) = \frac{\theta(t-\tau)}{(2a\sqrt{\pi(t-\tau)})^n} e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}}$$

is a fundamental solution of the operator $\frac{\partial}{\partial t} - a^2\Delta_x$ respect to the time variable, where θ is the Heviside function, and

$$U_3(x-\xi) = -\frac{1}{4\pi|x-\xi|}$$

is a fundamental solution of the 3-dimensional Laplacian $\Delta_x = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ in direction of perpendicular vectors on the boundary. Now, we claim that

$$U_2(x-\xi) = \frac{1}{2\pi} ln \sqrt{(x_1-\xi_1)^2 + |x_2-\xi_2|^2}$$

is fundamental solution of the 2-dimensional Laplace equation in direction of x_2. In fact,

$$\frac{\partial U_2(x-\xi)}{\partial x_2} = \frac{1}{2\pi}\frac{2|x_2-\xi_2|e(x_2-\xi_2)}{(x_1-\xi_1)^2+|x_2-\xi_2|^2},$$
$$\frac{\partial U_2(x-\xi)}{\partial x_1} = \frac{1}{2\pi}\frac{x_1-\xi_1}{(x_1-\xi_1)^2+|x_2-\xi_2|^2},$$
$$\frac{\partial^2 U_2(x-\xi)}{\partial x_2^2} = \frac{1}{\pi}\delta(x_2-\xi_2)\frac{|x_2-\xi_2|}{(x_1-\xi_1)^2+|x_2-\xi_2|^2}+ + \frac{1}{2\pi}\frac{1}{(x_1-\xi_1)^2+|x_2-\xi_2|^2}- - \frac{1}{\pi}\frac{|x_2-\xi_2|^2}{[(x_1-\xi_1)^2+|x_2-\xi_2|^2]^2},$$

and

$$\frac{\partial^2 U_2(x-\xi)}{\partial x_1^2} = \frac{1}{2\pi}\frac{-(x_1-\xi_1)^2+|x_2-\xi_2|^2}{[(x_1-\xi_1)^2+|x_2-\xi_2|^2]^2},$$

where e is the symmetric Heaviside function. Thus, $\Delta_2 U_2(x-\xi) = \delta(x-\xi)$. Also,

$$U_0(x-\xi) = \frac{\theta(x_2-\xi_2)}{2\pi}\int_0^\infty e^{-i\alpha_1(x_1-\xi_1)-\alpha_1(x_2-\xi_2)}\,\mathrm{d}\,\alpha_1 - \frac{\theta(\xi_2-x_2)}{2\pi}\int_0^\infty e^{i\alpha_1(x_1-\xi_1)-\alpha_1(\xi_2-x_2)}\,\mathrm{d}\,\alpha_1,$$

is fundamental solution of the Cauchy-Riemann operator $\frac{\partial}{\partial x_2}+i\frac{\partial}{\partial x_1}$, where θ is the Heviside function and $i^2=-1$ is the imaginary number, whereas some one give $\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}$ as fundamental solution of the Cauchy-Riemann equation ([18]). If $x_2 \neq \xi_2$, then two above solutions are same. But, the above solution U_0 is a better solution when $x_2 = \xi_2$.

Frequently, fractional differentiation of a function f be found by Fourier transform ([16]), whereas this method is not true. We will use the Cauchy formula for definition of fractional differentiation of a function, as did Liouville.

2. On the Problem A

By using Laplace transformation on (1) and (2), we have

$$\int_0^\infty e^{-\lambda t}\frac{\partial u_i(x,t)}{\partial t}\,\mathrm{d}\,t + \sum_{j=1}^{3}\int_0^\infty e^{-\lambda t}u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t =$$
$$= \nu\Delta\int_0^\infty e^{-\lambda t}u_i(x,t)\,\mathrm{d}\,t - \frac{\partial}{\partial x_i}\int_0^\infty e^{-\lambda t}p(x,t)\,\mathrm{d}\,t,$$

for all $i = 1, 2, 3$ and also,

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int_0^\infty e^{-\lambda t} u_i(x,t)\, \mathrm{d}\, t = 0,$$

where λ is a complex parameter and $\operatorname{Re}\lambda = c > 0$.

By using integration by part, we have

$$\int_0^\infty e^{-\lambda t} \frac{\partial u_i(x,t)}{\partial t}\, \mathrm{d}\, t = -u_i^0(x) + \lambda \int_0^\infty e^{-\lambda t} u_i(x,t)\, \mathrm{d}\, t.$$

Now, we show Laplace transforms of the functions u_i and p as following

$$\tilde{u_i}(x,\lambda) = \int_0^\infty e^{-\lambda t} u_i(x,t)\, \mathrm{d}\, t, \tag{8}$$

$$\tilde{p}(x,\lambda) = \int_0^\infty e^{-\lambda t} p(x,t)\, \mathrm{d}\, t. \tag{9}$$

Thus,

$$\Delta \tilde{u_i}(x,\lambda) - \frac{\lambda}{\nu} \tilde{u_i}(x,\lambda) = g_i(x,\lambda), \qquad (i = 1, 2, 3;\ x \in \mathbb{R}^3) \tag{10}$$

$$\sum_{j=1}^{3} \frac{\partial \tilde{u_j}(x,\lambda)}{\partial x_j} = 0, \qquad (x \in \mathbb{R}^3) \tag{11}$$

where

$$g_i(x,\lambda) = -\frac{1}{\nu} u_i^0(x) + \\ + \frac{1}{\nu} \sum_{j=1}^{3} \int_0^\infty e^{-\lambda t} u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j}\, \mathrm{d}\, t + \frac{1}{\nu} \frac{\partial}{\partial x_i} \tilde{p}(x,\lambda). \tag{12}$$

Left hand side of (10) is a Helmholts equation because left hand side coefficients of (10) are not depend on i. Hence by [18], its fundamental solution is

$$U(x-\xi,\lambda) = -\frac{e^{-\sqrt{\frac{\lambda}{\nu}}|x-\xi|}}{4\pi|x-\xi|}, \tag{13}$$

that is,

$$\Delta_x U(x-\xi,\lambda) - \frac{\lambda}{\nu} U(x-\xi,\lambda) = \delta(x-\xi), \tag{14}$$

where, $\lambda \in R_\sigma = \{\lambda : -\pi + \sigma \le \arg\lambda \le \pi - \sigma\}$ and we have fixed sufficiently small $\sigma > 0$.

Note that if $\lambda \in R_\sigma$, $|\lambda| \longrightarrow \infty$ and $x \neq \xi$, then $U(x-\xi,\lambda) \longrightarrow 0$.

Now, multiple (10) by $U(x-\xi,\lambda)$ and integrate the result on $\mathbb{R}^3$. Then,

$$\int_{\mathbb{R}^3} \Delta\tilde{u_i}(x,\lambda)U(x-\xi,\lambda)\,\mathrm{d}\,t - \frac{\lambda}{\nu}\int_{\mathbb{R}^3} \tilde{u_i}(x,\lambda)U(x-\xi,\lambda)\,\mathrm{d}\,x = \\ = \int_{\mathbb{R}^3} gi(x,\lambda)U(x-\xi,\lambda)\,\mathrm{d}\,x, \qquad (i=1,2,3)$$

By using Gauss-Ostrogradskii for the first integral of left hand side, from the last equation and by (7), we have

$$\int_{\mathbb{R}^3} \tilde{u_i}(x,\lambda)[\Delta_x U(x-\xi,\lambda) - \frac{\lambda}{\nu}U(x-\xi,\lambda)]\,\mathrm{d}\,x = \\ = \int_{\mathbb{R}^3} g_i(x,\lambda)U(x-\xi,\lambda)\,\mathrm{d}\,x.$$

Thus by (14),

$$\tilde{u_i}(\xi,\lambda) = \int_{\mathbb{R}^3} g_i(x,\lambda)U(x-\xi,\lambda)\,\mathrm{d}\,x, \quad (i=1,2,3;\ \xi\in\mathbb{R}^3). \tag{15}$$

By using the method in [1-3] and [10-13], we differentiate of the last equation. Then,

$$\frac{\partial\tilde{u_i}(\xi,\lambda)}{\partial\xi_k} = \int_{\mathbb{R}^3} g_i(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial\xi_k}\,\mathrm{d}\,t, \quad (i=1,2,3;\ \xi\in\mathbb{R}^3). \tag{16}$$

Now, by substituting (16) in (11), we obtain

$$0 = \sum_{i=1}^{3}\frac{\partial\tilde{u_i}(\xi,\lambda)}{\partial\xi_i} = \int_{\mathbb{R}^3}\sum_{i=1}^{3} g_i(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial\xi_i}\,\mathrm{d}\,x.$$

Substitute (12) in recent equation. Then,

$$\frac{-1}{\nu}\int_{\mathbb{R}^3}\sum_{i=1}^{3} u_i^0(x)\frac{\partial U(x-\xi,\lambda)}{\partial\xi_i}\,\mathrm{d}\,x+ \\ +\frac{1}{\nu}\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\frac{\partial U(x-\xi,\lambda)}{\partial\xi_i}dx\int_0^\infty e^{-\lambda t}u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t+ \\ +\frac{1}{\nu}\int_{\mathbb{R}^3}\sum_{i=1}^{3}\frac{\partial\tilde{p}(x,\lambda)}{\partial x_i}\frac{\partial U(x-\xi,\lambda)}{\partial\xi_i}\,\mathrm{d}\,x = 0.$$

By using (14), (7) and Gauss-Ostrogradskii formula ([7]) in the last expression of recent equation, we obtain

$$\frac{-1}{\nu}\int_{\mathbb{R}^3}\sum_{i=1}^{3}u_i^0(x)\frac{\partial U(x-\xi,\lambda)}{\partial \xi_i}\,\mathrm{d}\,x+ \\ +\frac{1}{\nu}\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\frac{\partial U(x-\xi,\lambda)}{\partial \xi_i}dx\int_0^\infty e^{-\lambda t}u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t+ \\ +\frac{1}{\nu}\int_{\mathbb{R}^3}\tilde{p}(x,\lambda)[\delta(x-\xi)+\frac{\lambda}{\nu}U(x-\xi,\lambda)]\,\mathrm{d}\,x=0.$$

Hence by property of Dirac's function, we have

$$\tilde{p}(\xi,\lambda)=\frac{-\lambda}{\nu}\int_{\mathbb{R}^3}\tilde{p}(x,\lambda)U(x-\xi,\lambda)\,\mathrm{d}\,x+ \\ +\int_{\mathbb{R}^3}\sum_{i=1}^{3}u_i^0(x)\frac{\partial U(x-\xi,\lambda)}{\partial \xi_i}\,\mathrm{d}\,x- \\ -\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\frac{\partial U(x-\xi,\lambda)}{\partial \xi_i}dx\int_0^\infty e^{-\lambda t}u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t. \tag{17}$$

Theorem 2.1. *Let $\nu > 0$ and u^0 satisfies (4). Then, solution of the nonlinear second type Fredholm system of equations with weak singularity (15) and (16), is a solution of (10).*

Theorem 2.2. *Let $\nu > 0$ and u^0 satisfies (4). Then, solution of the linear second type Fredholm equation with weak singularity (17), is obtained from (11).*

Remark 2.3. Note that the above method able us instead solving of system of equations (1), we first solve system of equations (15) and (16) and then by replacement of solutions in (17), we obtain a separate equation for $\tilde{p}$. By obtaining of $\tilde{p}$, one could obtain p. Also if $\lambda \in R_\sigma$ and $|\lambda| \longrightarrow \infty$, then $U(x-\xi,\lambda) \longrightarrow 0$. So the equation (17) could be solved easily.

Theorem 2.4. *If $\lambda \in R_\sigma$ and $|\lambda|$ is sufficiently large, then the equation (17) has an unique smoothness solution and this solution could be obtained by the successive approximation method.*

If as remark 2.3, we have a separate equation for $\tilde{p}$ from (17) and $H(x,\xi,\lambda)$ is the resolvent kernel of the equation, then

$$H(x,\xi,\lambda)=\sum_{k=0}^{\infty}\left(\frac{-\lambda}{\nu}\right)^k U_{k+1}(x,\xi,\lambda), \tag{18}$$

where,

$$U_1(x,\xi,\lambda)=U(x-\xi,\lambda), \tag{19}$$

and

$$U_{k+1}(x,\xi,\lambda)=\int_{\mathbb{R}^3}U_1(x,\gamma,\lambda)U_k(\gamma,\xi,\lambda)\,\mathrm{d}\,\gamma. \tag{20}$$

Note that, the series (18) is uniformly convergent whenever $\lambda\in R_\sigma$, $|\lambda|$ is sufficiently large and $x\neq\xi$. Also, treatments of the functions $H(x,\xi,\lambda)$ and $H(x-\xi,\lambda)$ are same ([15]). Thus,

$$\begin{aligned}\tilde{p}(\xi,\lambda)=&\\ =-\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\frac{\partial U(x-\xi,\lambda)}{\partial\xi_i}\,\mathrm{d}\,x\int_0^\infty e^{-\lambda t}u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t+&\\ +\frac{\lambda}{\nu}\int_{\mathbb{R}^3}H(x,\xi,\lambda)dx\int_{\mathbb{R}^3}\sum_{p,q=1}^{3}\frac{\partial U(\gamma-x,\lambda)}{\partial\xi_p}\,\mathrm{d}\,\gamma&\\ \int_0^\infty e^{-\lambda t}u_q(\gamma,t)\frac{\partial u_p(\gamma,t)}{\partial\gamma_q}\,\mathrm{d}\,t+\int_{\mathbb{R}^3}\sum_{i=1}^{3}u_i^0(x)\frac{\partial U(x-\xi,\lambda)}{\partial\xi_i}\,\mathrm{d}\,x-&\\ -\frac{\lambda}{\nu}\int_{\mathbb{R}^3}H(x,\xi,\lambda)\,\mathrm{d}\,x\int_{\mathbb{R}^3}\sum_{p=1}^{3}u_p^0(\gamma)\frac{\partial U(\gamma-x,\lambda)}{\partial x_p}\,\mathrm{d}\,\gamma.&\end{aligned} \tag{21}$$

By using (18), (19) and (20), we have

$$\frac{\partial U_1(x,\xi,\lambda)}{\partial x_1}=\frac{\partial U(x-\xi,\lambda)}{\partial x_1}=-\frac{\partial U(x-\xi,\lambda)}{\partial\xi_1}=-\frac{\partial U_1(x,\xi,\lambda)}{\partial\xi_1},$$

and in general,

$$\begin{aligned}\frac{\partial^{|\alpha|}U_1(x,\xi,\lambda)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}=\frac{\partial^{|\alpha|}U(x-\xi,\lambda)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}=&\\ =(-1)^{|\alpha|}\frac{\partial^{|\alpha|}U(x-\xi,\lambda)}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\partial\xi_3^{\alpha_3}}=(-1)^{|\alpha|}\frac{\partial^{|\alpha|}U_1(x,\xi,\lambda)}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\partial\xi_3^{\alpha_3}},&\end{aligned}$$

where, $\alpha_1,\alpha_2,\alpha_3$ are non-negative integers and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. Also,

$$\begin{aligned}\frac{\partial U_2(x,\xi,\lambda)}{\partial x_1} &= \frac{\partial}{\partial x_1}\int_{\mathbb{R}^3} U(x-\gamma,\lambda)U(\gamma-\xi,\lambda)\ \mathrm{d}\,\gamma = \\ &= \int_{\mathbb{R}^3} \frac{\partial U(x-\gamma,\lambda)}{\partial x_1} U(\gamma-\xi,\lambda)\ \mathrm{d}\,\gamma = \\ &= -\int_{\mathbb{R}^3} \frac{\partial U(x-\gamma,\lambda)}{\partial \gamma_1} U(\gamma-\xi,\lambda)\ \mathrm{d}\,\gamma = \\ &= \int_{\mathbb{R}^3} U(x-\gamma,\lambda)\frac{\partial U(\gamma-\xi,\lambda)}{\partial \gamma_1}\ \mathrm{d}\,\gamma = \\ &= -\int_{\mathbb{R}^3} U(x-\gamma,\lambda)\frac{\partial U(\gamma-\xi,\lambda)}{\partial \xi_1}\ \mathrm{d}\,\gamma = \\ &= -\frac{\partial}{\partial \xi_1}\int_{\mathbb{R}^3} U(x-\gamma,\lambda)U(\gamma-\xi,\lambda)\ \mathrm{d}\,\gamma = -\frac{\partial U_2(x,\xi,\lambda)}{\partial \xi_1}.\end{aligned}$$

In general,

$$\frac{\partial^{|\alpha|}U_2(x,\xi,\lambda)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} = (-1)^{|\alpha|}\frac{\partial^{|\alpha|}U_2(x,\xi,\lambda)}{\partial \xi_1^{\alpha_1}\partial \xi_2^{\alpha_2}\partial \xi_3^{\alpha_3}},$$

$$\frac{\partial^{|\alpha|}U_n(x,\xi,\lambda)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} = (-1)^{|\alpha|}\frac{\partial^{|\alpha|}U_n(x,\xi,\lambda)}{\partial \xi_1^{\alpha_1}\partial \xi_2^{\alpha_2}\partial \xi_3^{\alpha_3}}, \quad (n \geq 1), \tag{22}$$

and

$$\frac{\partial^{|\alpha|}H(x,\xi,\lambda)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} = (-1)^{|\alpha|}\frac{\partial^{|\alpha|}H(x,\xi,\lambda)}{\partial \xi_1^{\alpha_1}\partial \xi_2^{\alpha_2}\partial \xi_3^{\alpha_3}}. \tag{23}$$

Thus, we could rewrite two last expressions in right hand side of (21) as follows

$$\begin{aligned}&\int_{\mathbb{R}^3}\sum_{i=1}^{3} u_i^0(x)\frac{\partial U(x-\xi,\lambda)}{\partial \xi_i}\ \mathrm{d}\,x- \\ &\qquad -\frac{\lambda}{\nu}\int_{\mathbb{R}^3} H(x,\xi,\lambda)\ \mathrm{d}\,x\int_{\mathbb{R}^3}\sum_{p=1}^{3} u_p^0(\gamma)\frac{\partial U(\gamma-x,\lambda)}{\partial x_p}\ \mathrm{d}\,\gamma = \\ &\qquad\qquad = \int_{\mathbb{R}^3}\sum_{i=1}^{3}\frac{\partial u_i^0(x)}{\partial x_i}U(x-\xi,\lambda)\ \mathrm{d}\,x- \\ &\qquad\qquad -\frac{\lambda}{\nu}\int_{\mathbb{R}^3} H(x,\xi,\lambda)\ \mathrm{d}\,x\int_{\mathbb{R}^3}\sum_{p=1}^{3}\frac{\partial u_p^0(\gamma)}{\partial \gamma_p}U(\gamma-x,\lambda)\ \mathrm{d}\,\gamma.\end{aligned}$$

Note that, these two last expressions are of class C^∞ because one could transmit to $u_i^0(x)$ all partial differentiation of the functions $U(x-\xi,\lambda)$ and

$H(x,\xi,\lambda)$, until singularity will not appear. By replacement of (21) in (12), we find $g_i(x,\lambda)$ and then by replacement of the result in (15), we obtain

$$
\begin{aligned}
(24)\quad \tilde{u_i}(\xi,\lambda) = \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \Bigg\{ \frac{-1}{\nu} u_i^0(x) + \\
+ \frac{1}{\nu} \sum_{j=1}^{3} \int_0^\infty e^{-\lambda t} u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t + \\
+ \frac{1}{\nu}\frac{\partial}{\partial x_i} \Bigg[- \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} \frac{\partial U(\gamma - x,\lambda)}{\partial x_p}\,\mathrm{d}\,\gamma \int_0^\infty e^{-\lambda t} u_q(\gamma,t) \frac{\partial u_p(\gamma,t)}{\partial \gamma_q}\,\mathrm{d}\,t + \\
+ \frac{\lambda}{\nu} \int_{\mathbb{R}^3} H(\gamma,x,\lambda) d\gamma \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} \frac{\partial U(\zeta-\gamma,\lambda)}{\partial \gamma_p}\,\mathrm{d}\,\zeta \\
\int_0^\infty e^{-\lambda t} u_q(\zeta,t) \frac{\partial u_p(\zeta,t)}{\partial \zeta_q}\,\mathrm{d}\,t + \int_{\mathbb{R}^3} \sum_{p=1}^{3} u_p^0(\gamma) \frac{\partial U(\gamma - x,\lambda)}{\partial x_p}\,\mathrm{d}\,\gamma - \\
- \frac{\lambda}{\nu} \int_{\mathbb{R}^3} H(\gamma,x,\lambda)\,\mathrm{d}\,\gamma \int_{\mathbb{R}^3} \sum_{p=1}^{3} u_p^0(\zeta) \frac{\partial U(\zeta-\gamma,\lambda)}{\partial \gamma_p}\,\mathrm{d}\,\zeta \Bigg] \Bigg\} = \\
= \frac{-1}{\nu} \int_{\mathbb{R}^3} u_i^0(x) U(x-\xi,\lambda)\,\mathrm{d}\,x + \\
+ \frac{1}{\nu} \sum_{j=1}^{3} \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \int_0^\infty e^{-\lambda t} u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j}\,\mathrm{d}\,t - \\
- \frac{1}{\nu} \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} \frac{\partial U(\gamma - x,\lambda)}{\partial x_i}\,\mathrm{d}\,\gamma \\
\int_0^\infty e^{-\lambda t} \frac{\partial u_q(\gamma,t)}{\partial \gamma_p} \frac{\partial u_p(\gamma,t)}{\partial \gamma_q}\,\mathrm{d}\,t + \frac{\lambda}{\nu^2} \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \\
\int_{\mathbb{R}^3} \frac{\partial H(\gamma,x,\lambda)}{\partial x_i}\,\mathrm{d}\,\gamma \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} U(\zeta-\gamma,\lambda)\,\mathrm{d}\,\zeta \\
\int_0^\infty e^{-\lambda t} \frac{\partial u_q(\zeta,\lambda)}{\partial \zeta_p} \frac{\partial u_p(\zeta,t)}{\partial \zeta_q}\,\mathrm{d}\,t, \quad (i=1,2,3;\ \xi \in \mathbb{R}^3).
\end{aligned}
$$

Note that right hand side of expression of above equation is of class C^∞. Now for each $n \geq 0$, put

$$
(25)\qquad u_m^{(n)}(\xi,t) = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} e^{\lambda t} \tilde{u}_m^{(n)}(\xi,\lambda)\,\mathrm{d}\,\lambda,
$$

where, $m = 1, 2, 3$, $c > 0$, $\xi \in \mathbb{R}^3$, $u_m^{(0)} = u_m^0$, (n) is only a symbol and

$$(26) \quad \tilde{u}_m^{(n)}(\xi, \lambda) = \frac{-1}{\nu} \int_{\mathbb{R}^3} U(x - \xi, \lambda) u_m^0(x) \,\mathrm{d}\, x +$$

$$+ \frac{1}{\nu} \sum_{j=1}^{3} \int_{\mathbb{R}^3} U(x - \xi, \lambda) \,\mathrm{d}\, x \int_0^\infty e^{-\lambda t} u_j^{(n-1)}(x, t) \frac{\partial u_m^{(n-1)}(x, t)}{\partial x_j} \,\mathrm{d}\, t -$$

$$- \frac{1}{\nu} \int_{\mathbb{R}^3} U(x - \xi, \lambda) \,\mathrm{d}\, x \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} \frac{\partial U(\gamma - x, \lambda)}{\partial x_m} \,\mathrm{d}\, \gamma$$

$$\int_0^\infty e^{-\lambda t} \frac{\partial u_q^{(n-1)}(\gamma, t)}{\partial \gamma_p} \frac{\partial u_p^{(n-1)}(\gamma, t)}{\partial \gamma_q} \,\mathrm{d}\, t +$$

$$+ \frac{\lambda}{\nu^2} \int_{\mathbb{R}^3} U(x - \xi, \lambda) \,\mathrm{d}\, x \int_{\mathbb{R}^3} \frac{\partial H(\gamma, x, \lambda)}{\partial x_m} \,\mathrm{d}\, \gamma$$

$$\int_{\mathbb{R}^3} \sum_{p,q=1}^{3} U(\zeta - \gamma, \lambda) \,\mathrm{d}\, \zeta \int_0^\infty e^{-\lambda t} \frac{\partial u_q^{(n-1)}(\zeta, \lambda)}{\partial \zeta_p} \frac{\partial u_p^{(n-1)}(\zeta, t)}{\partial \zeta_q} \,\mathrm{d}\, t,$$

where $m = 1, 2, 3$, $n \geq 1$, $\xi \in \mathbb{R}^3$ and $\lambda \in R_\sigma$.

If $u_m^{(n-1)}(x, t)$ satisfies (6) and (7), for all $n \geq 1$ and $m = 1, 2, 3$, then $u_m^{(n)}(x, t)$ so is for all $n \geq 1$, because $\lambda \in R_\sigma$. Also by (25), $u_m^{(n)}(\xi, t)$ satisfies (6) and (7), for all $n \geq 1$ and $m = 1, 2, 3$. Since, every $\tilde{u}_m^{(n)}(\xi, \lambda)$ is of class C^∞ respect to ξ, so is $u_m^{(n)}(\xi, t)$. On the other hand, every $\tilde{u}_m^{(n)}(\xi, \lambda)$ is sufficiently small when $\lambda \in R_\sigma$ and $|\lambda| \longrightarrow \infty$, so $u_m^{(n)}(\xi, t)$ is of class C^∞ for all $n \geq 1$ and $m = 1, 2, 3$.

Remark 2.5. For arbitrary $d > 0$, put

$$\Gamma = \{c + it : -d \leq t \leq d\} \bigcup \{\lambda_1 + i\lambda_2 : \lambda_2 + d = tg\delta(\lambda_1 - c)\}$$
$$\bigcup \{\lambda_1 + i\lambda_2 : \lambda_2 - d = -tg\delta(\lambda_1 - c)\}.$$

Then, by Cauchy Theorem ([15])

$$\int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} e^{\lambda t} \tilde{u}_m^{(n)}(\xi, \lambda) \,\mathrm{d}\, \lambda = \int_\Gamma e^{\lambda t} \tilde{u}_m^{(n)}(\xi, \lambda) \,\mathrm{d}\, \lambda.$$

Since $\mathrm{Re}(\lambda) \longrightarrow -\infty$ in the right integral, the left integral exists. For $1 \leq m \leq 3$, define

$$T_m(\tilde{u}^{(n-1)}(\xi,\lambda)) = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} e^{\lambda t} \left\{ \frac{1}{\nu} \sum_{j=1}^{3} \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \right.$$

$$\int_0^\infty e^{-\lambda t} u_j^{(n-1)}(x,t) \frac{\partial u_m^{(n-1)}(x,t)}{\partial x_j}\,\mathrm{d}\,t -$$

$$-\frac{1}{\nu} \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} \frac{\partial U(\gamma - x,\lambda)}{\partial x_m}\,\mathrm{d}\,\gamma$$

$$\int_0^\infty e^{-\lambda t} \frac{\partial u_q^{(n-1)}(\gamma,t)}{\partial \gamma_p} \frac{\partial u_p^{(n-1)}(\gamma,t)}{\partial \gamma_q}\,\mathrm{d}\,t +$$

$$+\frac{1}{\nu^2} \int_{\mathbb{R}^3} U(x-\xi,\lambda)\,\mathrm{d}\,x \int_{\mathbb{R}^3} \frac{\partial H(\gamma,x,\lambda)}{\partial x_m}\,\mathrm{d}\,\gamma$$

$$\left. \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} U(\zeta-\gamma,\lambda)\,\mathrm{d}\,\zeta \int_0^\infty e^{-\lambda t} \frac{\partial u_q^{(n-1)}(\zeta,\lambda)}{\partial \zeta_p} \frac{\partial u_p^{(n-1)}(\zeta,t)}{\partial \zeta_q}\,\mathrm{d}\,t \right\}.$$

Then, by (25)

$$u_m^{(n)}(\xi,t) = T_m(u^{(n-1)}(\xi,\lambda)) + h_m(\xi,t), \tag{27}$$

where,

$$h_m(\xi,t) = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} e^{\lambda t} \left\{ \frac{-1}{\nu} \int_{\mathbb{R}^3} U(x-\xi,\lambda) u_m^0(x)\,\mathrm{d}\,x \right\}.$$

Finally, suppose that S is the set of all functions satisfy (6) and (7).

Theorem 2.6. *The transformation $T_m : S \longrightarrow S$ is a contraction and the sequence $\{u_m^{(n)}\}_{n\geq 1}$ is convergent for all $m = 1,2,3$. If $u_m^{(n)} \longrightarrow u_m$ (m=1,2,3), then $u = (u_1,u_2,u_3)$ is the unique solution of the problem A.*

Now, if we put in (21) these u_1, u_2 and u_3, we could find $\tilde{p}(\xi,\lambda)$ and so we could find

$$p(x,t) = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} e^{\lambda t} \tilde{p}(x,\lambda)\,\mathrm{d}\,\lambda.$$

Therefore, the problem A be solved completely.

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